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Some Properties of Fourier Strips, with Applications to the Digital Computer

BY J. MONTEATH ROBERTSON

Chemistry Department, The University, Glasgow W. 2, Scotland

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By a simple modification the number of figures which it is necessary to print in the usual Beevers–Lipson type of Fourier strip may be reduced to one quarter, if the summation interval is correctly chosen. This modification may be useful if it is desired to produce strips for finer axial sub-division, and it has obvious applications in simplifying the digital computer previously described.

Introduction

The Fourier-strip method described by Lipson & Beevers (1936) was originally designed for summations at 60ths of the cell edge, but in the new strips recently produced by Beevers (1952) this interval is decreased to 120ths. In Beevers' method separate strips are provided for each value of the index h, for sines and cosines, and for positive and negative coefficients. In the alternative method described by Robertson (1936) there is only one strip for each coefficient. In these strips the interval of the cell edge sub-division has also been decreased to 120ths (Robertson, 1948).

This note deals with a property possessed by the former kind of strip which does not appear to have been noted, and which reduces the number of figures which it is necessary to print. The arrangement may be useful if it is desired to produce strips of still smaller cell-edge sub-division, and it also has obvious applications in reducing the complexity or extending the scope of the digital machine recently described (Robertson, 1954).

Applications to Fourier strips

This property arises from the nature of the four matrices given in Table 2 of the previous paper (Robertson, 1954). The figures in the body of this table represent the angles (in degrees) whose sines are required in effecting any Fourier-series summation. These matrices possess certain symmetries which enable them to be rearranged in various ways. It is clearly not necessary to employ all these separate factors (or sine and cosine generators in the case of the digital machine) for a Fourier-series summation. In the limiting case it is only necessary to employ a single row for $h = 1$, as the sines of these angles include all the different factors that are necessary. Some form of sorting device is then necessary, and this is in fact the form of figure strip method described by Robertson (1936). The arrangement presented in the complete table is actually that employed in the Beevers–Lipson strips, and in the digital machine already described. It has the advantage of eliminating the need for any form of sorting device, or switching arrangement in the case of the machine. This advantage is gained at the expense of a larger number of strips (about 12000 instead of 100 for two-figure work at 3° intervals).

Between these two extremes various intermediate arrangements are possible which reduce the number of separate factors required. In particular, there is one very simple way in which the number of factors required may be reduced to one quarter, with only very slight additional complication. This reduction can be applied either to the Beevers–Lipson type of Fourier strip, when it becomes necessary to print only one quarter of the number of terms previously employed, or to the digital machine already described, when only one quarter of the number of generators is required, and half the number of counters and differentials.

For this modification to be effective it is a necessary condition that if N is the number of axial sub-divisions at which the summation is first effected over the completed period, then N should be even but $N/2$ odd. The existing strips, designed for summations at 60ths and 120ths do not, of course, conform to this requirement.

Consider as an example the sine factors required (Table 1) for the strip $h = 2$ for summations at 30ths

Table 1

	0°	12°	24°	36°	48°	60°	72°	84°
$h = 2$	0	0·407	0·743	0·951	0·995	0·866	0·588	0·208
	90°	78°	60°	54°	42°	30°	18°	6°

(intervals of 12°). Because the number of axial sub-divisions in this case is not divisible by 4 ($N/2$ odd) the distribution of points at which the summation is made straddles the quarter-point at 90°. Consequently the figures on the strip, reading from right to left, are also the factors required for summations at the intermediate points 6°, 18°, ..., 90° which are shown under the sine factors in the above example.

This relation holds for all the sine strips with $h = 2, 6, 10, \ldots$. For $h = 4, 8, 12, \ldots$ the sign of the factors must be reversed (or the strip turned over, if negative values are printed on the back) in order to obtain summations at the intermediate points. For the even cosine strips similar relations hold, the sign reversals in this case occurring for the strips with $h = 2, 6, 10, \ldots$. In general, if the interval between two successive summation points is θ, expressed in degrees, and n is the number of the point starting from the origin, so that n has the values $0, 1, 2, \ldots, (N-1)$ over a complete period, then if $N/2$ is odd, $(90-n\theta)$ always gives a new intermediate point lying half-way between those of the initial summation. We confine ourselves to points lying in the first quadrant, as is the general practice in Fourier-strip methods, the remaining points required to complete the period being obtained from the symmetry of the functions, employing the usual p, q, r, s relations (Robertson, 1954).

Now, we have the relations

$$
\begin{aligned}
\cos h(90-n\theta) &= \cos hn\theta \quad \text{when} \quad h = 4, 8, 12, \ldots\\
\cos h(90-n\theta) &= -\cos hn\theta \quad \text{when} \quad h = 2, 6, 10, \ldots\\
\cos h(90-n\theta) &= \sin hn\theta \quad \text{when} \quad h = 1, 5, 9, \ldots\\
\cos h(90-n\theta) &= -\sin hn\theta \quad \text{when} \quad h = 3, 7, 11, \ldots\\
\sin h(90-n\theta) &= \cos hn\theta \quad \text{when} \quad h = 1, 5, 9, \ldots\\
\sin h(90-n\theta) &= -\cos hn\theta \quad \text{when} \quad h = 3, 7, 11, \ldots\\
\sin h(90-n\theta) &= \sin hn\theta \quad \text{when} \quad h = 2, 6, 10, \ldots\\
\sin h(90-n\theta) &= -\sin hn\theta \quad \text{when} \quad h = 4, 8, 12, \ldots
\end{aligned}
$$

which explain the sign inversions for the different strips mentioned above. We also note that when h is odd sine strips must be used instead of cosine strips, and vice versa. However, as h and n are interchangeable in the above relations it is obvious that intermediate values of the index h can be obtained in a precisely analogous manner to that employed for obtaining intermediate values of n. Hence, strips for odd values of the index h are unnecessary, as these can all be obtained from the strips with even values.

As an example, the only strips now necessary for summations at 60ths of the cell edge ($\theta = 6°$) are shown in Table 2. The notation here is the same as that of Table 2 of the previous paper (Robertson, 1954), the factors in the body of the table being represented by the angles, in degrees, whose sines are required. For two-figure work there would, of course, be 100 positive and 100 negative coefficients for each row in the table.

To effect Fourier summations with strips prepared in this way it will usually be most convenient to carry out the work in two stages, the first summation being for the basic interval for which the strip is designed (in this example 12° or 30ths). A second summation can then be carried out if required for the intermediate points (giving 6° intervals or 60ths in this example). The first summation is carried out in the usual way, except that as we have dispensed with the odd cosine (q) and odd sine (r) strips, the even cosine (p) and even sine (s) strips are employed again, as indicated in the table, with sign reversals for the *columns* marked #. (It is, of course, not necessary to reverse the signs of alternate numbers on individual strips, which would be awkward, but only the signs of the totals obtained after addition.)

Table 2

Cosines

			0°	12°	24°	36°	48°	60°	72°	84°	
			90°	78°	66°	54°	42°	30°	18°	6°	*
	p	q									
	0	15	90	90	90	90	90	90	90	90	
	2	13	90	66	42	18	$\overline{6}$	$\overline{30}$	$\overline{54}$	$\overline{78}$	*
	4	11	90	42	$\overline{6}$	$\overline{54}$	$\overline{78}$	$\overline{30}$	18	66	
	6	10	90	18	$\overline{54}$	$\overline{54}$	18	90	18	$\overline{54}$	*
h	8	7	90	$\overline{6}$	$\overline{78}$	18	66	$\overline{30}$	$\overline{54}$	42	
	10	5	90	$\overline{30}$	$\overline{30}$	90	$\overline{30}$	$\overline{30}$	90	$\overline{30}$	*
	12	3	90	$\overline{54}$	18	18	$\overline{54}$	90	$\overline{54}$	18	
	14	1	90	$\overline{78}$	66	$\overline{54}$	42	$\overline{30}$	18	$\overline{6}$	*
			#		#		#		#		

Sines

			0°	12°	24°	36°	48°	60°	72°	84°	
			90°	78°	66°	54°	42°	30°	18°	6°	*
	s	r									
	0	15	.	.	.	.	.	.	.	.	
	2	13	.	24	48	72	84	60	36	12	
	4	11	.	48	84	36	$\overline{12}$	$\overline{60}$	$\overline{72}$	$\overline{24}$	*
	6	9	.	72	36	$\overline{36}$	$\overline{72}$	.	72	36	
h	8	7	.	84	$\overline{12}$	$\overline{72}$	24	60	$\overline{36}$	$\overline{48}$	*
	10	5	.	60	$\overline{60}$	.	60	$\overline{60}$	.	60	
	12	3	.	36	$\overline{72}$	72	$\overline{36}$	.	36	$\overline{72}$	*
	14	1	.	12	$\overline{24}$	36	$\overline{48}$	60	$\overline{72}$	84	
			#		#		#		#		

If a second summation for the intermediate points is now required, the same strips are used again. All the even-index terms are dealt with in the usual way, except that there are now sign reversals for the rows (strips) marked * in the table. This can be effected by turning over these strips, if negative values are printed on the back. For the odd-index cosine terms we now draw the appropriate *sine* strips, and reverse signs for both the rows and columns marked in the table. Similarly, for the odd-index sine terms we draw the appropriate *cosine* strips, reverse the signs for the columns marked in the table, and for the rows which are *not* marked.

If the work is carried out systematically, which is always necessary in any case if mistakes are to be avoided, there should be no difficulty in observing these rules. The strips might also be labelled in a manner that would facilitate the transpositions required.

The method for dealing with terms having indices in the range $h = 16$–30 is now actually much simplified as compared with the ordinary strip method and the method outlined for the digital computer (Robertson, 1954), because the double summation has the effect of separating the odd and even ordinates (values of n). As a result, the coefficients of all terms in the range $h = 16$–30 can now be combined with the coefficients of the terms in the range $h = 0$–15 before commencing the summations. The rules are as follows. For the first summation over the points (ordinates) 0°, 12°, 24°, ..., 84°, the coefficients of the cosine terms with indices in the range $h = 16$–30 are added to the coefficients of the terms with index $(30-h)$ if h is even, or to the terms with $(h-15)$ if h is odd. The coefficients of the sine terms in the range $h = 16$–30 are subtracted from the coefficients of the sine terms with index $(30-h)$ (h even) or $(h-15)$ (h odd). For the second summation over the points (ordinates) 6°, 18°, 30°, ..., 90°, reading from the right, this process is reversed, i.e. the appropriate sine coefficients are added, and the cosine coefficients are subtracted. After the coefficients have been combined in this way, the summations are then carried out as described above.

It has already been noted that the existing strips, which are designed for summations at 60ths and 120ths of the axial lengths, do not conform to the requirement that $N/2$ should be odd, and so they cannot be used to produce summations at intermediate points. It is not suggested that these strips should now be changed, but sub-division at some other interval might have been more useful. For example, an interval of 7·2° with 13 entries on each strip (or 12 in the case of sine terms) would provide for summations at 50ths of the cell edge direct, and at 100ths by means of a second summation according to the above scheme. Such strips would be nearly as useful as the present ones which give 120ths. If a finer sub-division were ever required an interval of 4° might be suggested, with 23 entries on the strip. Direct summation would give 90ths and a second summation 180ths. Finally, a useful decimal sub-division which might have applications in structure-factor work could be provided by an interval of 1·44° with 63 entries on the strip. The first summation in this case would give 250ths, and the second 500ths of the cell edge.

Applications to the digital computer

These results have a useful application in the design of the digital computer already described (Robertson, 1954). Instead of the four banks of generators described in that paper it will be sufficient to employ only the two smaller banks based on Table 2. With this modification it will no longer be possible to feed in coefficients corresponding to odd and even index values simultaneously, but simultaneous operation of the sine and cosine banks will still be possible. Negative values of the coefficients for the various rows of generators can, of course, be fed in as easily as positive values, and the controls can be labelled appropriately. The sign reversals required for alternate columns of generators can be built into the totalizer units.

For summations at intermediate points the machine would then be run through a second cycle of operations. During this second cycle, the function of the second and fourth output counters for each column (Table 2) would be interchanged, the second counter yielding the sum at $360°-n\theta$, and the fourth counter the sum at $180°-n\theta$, because of the interchange of the sine and cosine generators for the odd-index coefficients (q and r). This would not cause any complication, the output counters being labelled accordingly.

As there are now only two banks of generators instead of four, it would be possible to simplify the totalizer units by dispensing with the coupled pairs of differentials at each end of these units, retaining only the four differentials immediately below the four output shafts. There would then be only two input shafts dealing with the cosine terms (p or q) and with the sine terms (r or s) respectively, each of these shafts being coupled to each of the remaining four output differentials. Although perhaps simpler, the disadvantage of this arrangement would be the need for reverse gears on four of the shafts, to be operated when odd-index terms are employed.

It is probably better to retain the totalizer units as already described (Robertson, 1954), switching the input from p to q or from s to r by means of a further differential at each end of the unit. This would eliminate any need for reverse gears and retain all the advantages of automatic operation. The sign reversals required for alternate columns of generators can be built in as a permanent feature on the q and r shafts.

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